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1994 J. Phys. A: Math. Gen. 27 7201

(http://iopscience.iop.org/0305-4470/27/21/034)

J. Phys. A: Math. Gen. 27 (1994) 7201–7208. Printed in the UK

Mean value and uncertainty of optical phase—a simple mechanical analogy

Tomáš Opatrný†

Department of Theoretical Physics, Palacký University, Svobody 26, 771 46 Olomouc, Czech Republic

Received 21 June 1994, in final form 7 September 1994

Abstract. There exist several definitions of the mean value and the phase uncertainty of an optical field. We show that one of these definitions is especially simple and intuitive because of its mechanical analogy. On this basis, we also derive the summation rule and the Tchebyshev inequality for random angular and phase variables and a number–phase uncertainty relation.

1. Introduction

In quantum optics, there has been much discussion about the measurement of the phase of an optical-field. Along with the problem of how to define an operator of this quantity, it is not clear how to define its mean value and spread (uncertainty). If we are able to construct some phase distribution $p(\varphi)$, where $p(\varphi+2\pi)=p(\varphi)$ and $\int_0^{2\pi} p(\varphi)\,\mathrm{d}\varphi = 1$, then there are several ways of defining these characteristics of φ. Similarly, as for a real-axis random variable, we could be tempted to take a mean value defined as

$$\langle\varphi\rangle_\beta = \int_\beta^{\beta+2\pi} \varphi p(\varphi)\,\mathrm{d}\varphi \tag{1}$$

and choose the variance

$$D_\beta(\varphi) = \int_\beta^{\beta+2\pi} (\varphi - \langle\varphi\rangle_\beta)^2 p(\varphi)\,\mathrm{d}\varphi \tag{2}$$

as the uncertainty. However, this definition is dependent on the chosen phase interval: if we change the origin of the phase window β, the mean value and variance change. Some authors use realizations of phase uncertainty by choosing an interval for which the variance has a minimum (e.g. [1]; measurements of the quantity D_β see, e.g., [2]).

The aim of this paper is to show that the definition of the preferred phase [1]

$$\bar\varphi = \arg\langle\exp(\mathrm{i}\varphi)\rangle = \arg\left(\int_\beta^{\beta+2\pi} \exp(\mathrm{i}\varphi) p(\varphi)\,\mathrm{d}\varphi\right) \tag{3}$$

and that of the dispersion (first introduced into quantum optics in [3], studied in the classical theory of statistics in [4] and discussed, for example, in [1, 5–7])

$$\sigma_\varphi^2 = 1 - \langle\cos\varphi\rangle^2 - \langle\sin\varphi\rangle^2 \tag{4}$$

† E-mail: opatrny@risc.upol.cz

have a simple mechanical analogy, which provides a deeper insight into the phase distribution. Moreover, we propose a new measure of the phase uncertainty $\Delta\varphi = \sin^{-1}\sigma_\varphi$ with an interpretation close to the ordinary standard deviation [8].

To distinguish the ordinary mean value and uncertainty from their angular and phase counterparts, in this paper we use the symbols $\langle\varphi\rangle$ and D to denote the usual phase mean and variance, respectively (equations (1) and (2); the origin of the phase window β is chosen so that D_β is minimized), whereas $\bar{\varphi}$ and σ_φ^2 denote the preferred phase and the dispersion (equations (3) and (4), respectively). The symbol $\Delta_0\varphi$ is used for the usual definition of the phase standard deviation $\Delta_0\varphi = \sqrt{D}$, while $\Delta\varphi$ is used for the proposed measure of uncertainty.

This paper is organized as follows. In section 2, we study the analogy between the mechanical and probabilistic variables to gain better insight into the problem. In sections 3 and 4, we modify some results of the probability theory for the case of angular variables, concerning the summation of random variables (section 3) and the Tchebyshev inequality (section 4). Finally, in section 5, we apply the new results to the optical phase and give some simple examples.

2. Probability–mass analogy

It is useful to note that the mechanical analogy of the mean value is the centre of mass. Let us consider, for simplicity, a two-dimensional body with unit mass, its density being described by the function $\rho(x, y)$. The coordinates of its centre of mass C are given by $x_c = \int x\rho(x, y)\,dx\,dy$, $y_c = \int y\rho(x, y)\,dx\,dy$. This is formally the same expression as those for the mean values of random variables x and y. In contrast, the quantity $I = \int \delta^2(x, y)\rho(x, y)\,dx\,dy$, where $\delta(x, y)$ is the distance of the varying point from the centre of mass, is just the moment of inertia (with respect to the centre of mass and the axis perpendicular to the plane xy), which is a simple analogy of $\mathrm{Tr}\,A_\Delta$, where

$$A_\Delta = \begin{pmatrix} \langle(\Delta x)^2\rangle & \langle\Delta x \Delta y\rangle \\ \langle\Delta y \Delta x\rangle & \langle(\Delta y)^2\rangle \end{pmatrix}. \tag{5}$$

This covariance matrix corresponds to the inertia tensor.

We can use this analogy to interpret the preferred angle and the dispersion of angle. In our analogy, we will suppose that the angle distribution corresponds to the density of a ring. Let us consider a ring with unit radius and unit mass, its density being described by a function of angle $\rho(\varphi)$. The centre of mass can be described by the polar coordinates R and $\bar{\varphi}$, which are given implicitly by

$$R\cos\bar{\varphi} = \int_0^{2\pi} \cos\varphi\,\rho(\varphi)\,d\varphi \tag{6}$$

$$R\sin\bar{\varphi} = \int_0^{2\pi} \sin\varphi\,\rho(\varphi)\,d\varphi. \tag{7}$$

Note that the explicit definition of $\bar{\varphi}$ coincides with equation (3).

Now we can compute the moment of inertia with respect to the centre of mass (along the axis perpendicular to the plane of the ring). We write

$$I = \int_0^{2\pi} \delta^2(\varphi)\rho(\varphi)\,d\varphi \tag{8}$$

where the $\delta(\varphi)$ is the Euclidean distance between the centre of mass and the point of the ring with angular coordinate φ, i.e. (using the cosine theorem)

$$\delta^2(\varphi) = 1 + R^2 - 2R\cos(\varphi - \bar{\varphi}). \tag{9}$$

Computing the integral (equation (8)), we find that

$$I = 1 - R^2. \tag{10}$$

We can easily verify that this quantity is equivalent to the phase dispersion defined by equation (4).

This result may be used for defining the uncertainty of φ (see figure 1). Let us consider the quantity $\sigma = I^{1/2}$; for a unit-mass body with a centre of mass C, it is a length with the following meaning: if we place two half-mass points in the plane of the body at a distance σ from the point C, so that C is in the middle between them, we get a body with the same centre of mass C and the same moment of inertia I. Locating these two points on the unit circle, the angular distance between them may be used as a measure of the angular uncertainty. We can write the angular coordinates of these points

$$\varphi_{1,2} = \bar{\varphi} \pm \Delta\varphi \tag{11}$$

where

$$\Delta\varphi = \cos^{-1}(R). \tag{12}$$

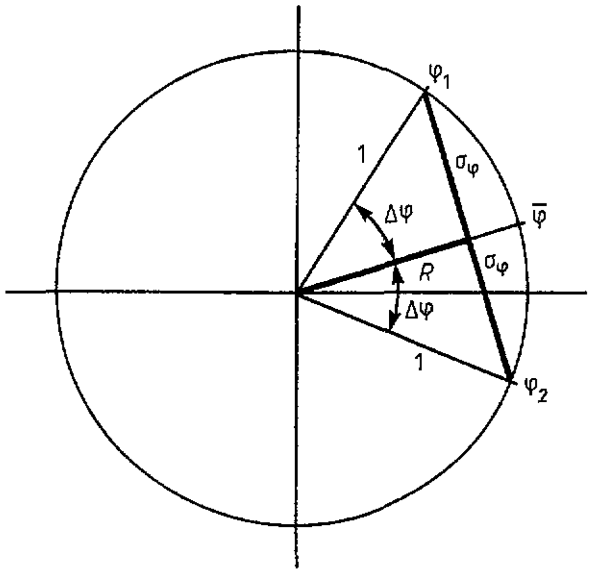

Figure 1. Two half-mass points located at φ_1 and φ_2 on the unit circle have the centre of mass with polar coordinates R and $\bar{\varphi}$, thus defining the preferred angle $\bar{\varphi}$ and the angular uncertainty $\Delta\varphi$ through the probability–mass analogy.

If we replace $\rho(\varphi)$ by $p(\varphi)$ in equations (6)–(8) and interpret it as the distribution of the angle φ, the preferred angle $\bar{\varphi}$ can be treated as a uniquely defined mean value of the angle and then the quantity $\Delta\varphi$ can measure the angular uncertainty; equation (12) provides the values in the interval $[0; \pi/2]$. Let us note that in the case of $R = 0$, $\bar{\varphi}$ is undefined.

3. Summation of random angle variables

In the Euclidean space theory of probability, the expectation (mean) and the variance of a sum of independent random variables have simple expressions. Let X and Y be two independent random Euclidean variables. Then, the expectation of their sum is $E(X+Y) = E(X) + E(Y)$ and the variance of their sum is $D(X+Y) = D(X) + D(Y)$. We can modify these rules for the random angular variables.

Writing (6) and (7) in a simpler form

$$R\,\mathrm{e}^{\mathrm{i}\bar{\varphi}} = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\varphi} p(\varphi)\,\mathrm{d}\varphi \tag{13}$$

we will assume that $\varphi = \chi + \vartheta$. Here χ and ϑ are independent angular variables with phase distributions $p_\chi(\varphi)$ and $p_\vartheta(\varphi)$, respectively, for which we can write $R_\chi \exp(\mathrm{i}\bar{\chi}) = \langle \exp(\mathrm{i}\chi)\rangle$ and $R_\vartheta \exp(\mathrm{i}\bar{\vartheta}) = \langle \exp(\mathrm{i}\vartheta)\rangle$. Then we can write

$$R\,\mathrm{e}^{\mathrm{i}\bar{\varphi}} = \langle \mathrm{e}^{\mathrm{i}(\chi+\vartheta)}\rangle = \int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\chi}\mathrm{e}^{\mathrm{i}\vartheta} p_\chi(\chi) p_\vartheta(\vartheta)\,\mathrm{d}\chi\,\mathrm{d}\vartheta = R_\chi \mathrm{e}^{\mathrm{i}\bar{\chi}} R_\vartheta \mathrm{e}^{\mathrm{i}\bar{\vartheta}} \tag{14}$$

from which we conclude that

$$R = R_\chi R_\vartheta \tag{15}$$

and

$$\bar{\varphi} \equiv \bar{\chi} + \bar{\vartheta} \bmod 2\pi. \tag{16}$$

For the preferred values, the summation rule is very similar to that of Euclidean expectation, whereas, for the dispersions, the situation is different from the variances. We can write

$$\sigma_\varphi^2 = 1 - R^2 = 1 - R_\chi^2 R_\vartheta^2 = \sigma_\chi^2 + \sigma_\vartheta^2 - \sigma_\chi^2\sigma_\vartheta^2. \tag{17}$$

This summation rule for the dispersions differs from that for the variances of Euclidean variables by the term $-\sigma_\chi^2\sigma_\vartheta^2$; however, we can see that they give similar results for σ_χ^2, $\sigma_\vartheta^2 \ll 1$.

Let us mention an interesting example of the summation of angular variables. We can consider a sum $\phi = \sum_{k=1}^N \varphi_k$ of N independent random angular variables φ_k, each with the dispersion σ_0^2/N. In the limit of N tending to infinity, the radial value R_ϕ of the sum tends to the value

$$R_\infty = \lim_{N\to\infty} R_\phi = \lim_{N\to\infty} (1 - \sigma_0^2/N)^{N/2} = \mathrm{e}^{-\sigma_0^2/2} \tag{18}$$

and the dispersion tends to

$$\sigma_\infty^2 = 1 - R_\infty^2 = 1 - \mathrm{e}^{-\sigma_0^2}. \tag{19}$$

Again, this result differs from that of the Euclidean variables, where σ_∞^2 would be equal to σ_0^2, but the results approach each other when $\sigma_0^2 \ll 1$. We can also make a conclusion about the distribution of the summed angle ϕ. If the summed variables were Euclidean, we would get the normal distribution with variance σ_0^2, according to the central-limit theorem. Due to the periodicity of the angular distribution, we arrive at the so-called wrapped normal distribution [4]

$$p(\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos(n(\phi - \bar{\phi}))\exp(-n^2\sigma_0^2/2). \tag{20}$$

As may be checked, the dispersion of this distribution agrees with equation (19).

4. Tchebyshev inequality

The Tchebyshev inequality is a very important theorem of probability theory. It can generally be written in the form

$$P(|X - \bar{X}| > \epsilon) \leqslant \frac{\sigma_X^2}{\epsilon^2} \tag{21}$$

where ϵ is an arbitrary positive real number, X is a random variable with mean $\bar{X}$ and variance σ_X^2 and P denotes the probability of the event in the parentheses.

A similar inequality can also be derived for the case of angular distribution. The derivation is essentially the same as the derivation of the usual form of the Tchebyshev inequality [9]. Let us write the dispersion, as in (8) and (9), with changed integration limits

$$\sigma_\varphi^2 = \int_{\bar{\varphi}-\pi}^{\bar{\varphi}+\pi} (1 + R^2 - 2R\cos(\varphi - \bar{\varphi}))p(\varphi)\,\mathrm{d}\varphi. \tag{22}$$

Let ε be a real number between zero and π; the integral (22) can then be written as a sum of three integrals

$$\sigma_\varphi^2 = \left[\int_{\bar{\varphi}-\pi}^{\bar{\varphi}-\varepsilon} + \int_{\bar{\varphi}-\varepsilon}^{\bar{\varphi}+\varepsilon} + \int_{\bar{\varphi}+\varepsilon}^{\bar{\varphi}+\pi}\right](1 + R^2 - 2R\cos(\varphi - \bar{\varphi}))p(\varphi)\,\mathrm{d}\varphi. \tag{23}$$

Since the integrand is always non-negative, the right-hand side will not increase if the middle integral is dropped. Then, because, in both remaining intervals, $\cos(\varphi - \bar{\varphi}) \leqslant \cos\varepsilon$, we may write

$$\begin{aligned}\sigma_\varphi^2 &\geqslant (1 + R^2 - 2R\cos\varepsilon)\left[\int_{\bar{\varphi}-\pi}^{\bar{\varphi}-\varepsilon} + \int_{\bar{\varphi}+\varepsilon}^{\bar{\varphi}+\pi}\right]p(\varphi)\,\mathrm{d}\varphi \\ &= (1 + R^2 - 2R\cos\varepsilon)P(|\varphi - \bar{\varphi}|\rangle\varepsilon)\end{aligned} \tag{24}$$

assuming that $|\varphi - \bar{\varphi}| \leqslant \pi$. Thus, we can write

$$P\left(|\varphi - \bar{\varphi}| > \varepsilon\right) \leqslant \frac{1 - R^2}{1 + R^2 - 2R\cos\varepsilon} \tag{25}$$

or, introducing $\Delta\varphi$,

$$P\left(|\varphi - \bar{\varphi}| > \varepsilon\right) \leqslant \frac{\sin^2(\Delta\varphi)}{1 + \cos^2(\Delta\varphi) - 2\cos(\Delta\varphi)\cos\varepsilon} \tag{26}$$

for any $\varepsilon \in (0, \pi)$. This is the Tchebyshev inequality for the angular distribution: it shows the relationship between the notion of angular uncertainty $\Delta\varphi$ and the probability of an angle falling outside some interval centred at $\bar{\varphi}$.

It can easily be shown that, in the case of $\Delta\varphi \ll 1$ and $\varepsilon \ll 1$, the right-hand side of the last inequality approaches $(\Delta\varphi)^2/\varepsilon^2$, according to the usual form of the Tchebyshev inequality.

It is also worth noting that the only case for which the inequality becomes an equality is when the angular distribution has the form of two equal delta peaks, located at $\bar{\varphi} + \Delta\varphi$ and $\bar{\varphi} - \Delta\varphi$, and when $\varepsilon = \Delta\varphi$.

5. Optical phase

Up to now the analogy between probability and mass has been exploited. To recognize the analogy between angle and phase, let us remember that the phase space of a particle moving in a plane has four components x, y, p_x and p_y. From the adopted static point of view, the angle is determined by the coordinates x and y. In contrast, the phase space of an optical mode is similar to that of a one-dimensional harmonic oscillator with its two coordinates x and p_x. *Thus, the phase is a dynamical concept, which is defined in terms of x and p_x.* The analysis of the preferred angle and the angular dispersion applies also to the preferred phase and the phase dispersion because of the formal replacement of x and y by x and p_x according to the standard correspondence [10]

$$x \to \sqrt{\frac{m\omega}{2\hbar}}x \to \mathrm{Re}\,\alpha \tag{27}$$

$$y \to \frac{1}{\sqrt{2\hbar m\omega}}p_x \to \mathrm{Im}\,\alpha \tag{28}$$

where ω is the frequency of the harmonic oscillator and m is the mass of the particle. Finally, α is the complex amplitude of the optical mode.

For comparison, we calculate the phase uncertainty using both the usual definition and the proposed method for two simple cases.

5.1. Uniformly distributed phase

Suppose that the phase distribution has the form $p(\varphi) = (1/N)\sum_{k=1}^{N}\delta(\varphi - 2\pi k/N)$; it may correspond, for example, to any number state in the Pegg–Barnett model [11], where N is the dimension of the Hilbert space. The usual definition of the uncertainty (equation (2)) gives the N-dependent value

$$\Delta_0\varphi = \frac{\pi}{\sqrt{3}}\sqrt{1 - \frac{1}{N^2}} \tag{29}$$

which, in the limit of infinite N (and for the case $p(\varphi) = 1/(2\pi)$), gives the well known value [1, 5] $\Delta_0\varphi = \pi/\sqrt{3}$. The proposed method for measuring the phase uncertainty gives the same result for every $N \geqslant 2$: $\Delta\varphi = \pi/2$. It can easily be understood by using our analogy if we realize that, for this phase distribution, the centre of mass coincides with the centre of the ring ($R = 0$); the moment of inertia would then be reproduced by a phase distribution of two delta peaks located at $\pm\pi/2$ from the arbitrary initial direction.

5.2. Superposition of two phase states

Let the phase distribution have the form

$$p(\varphi) = p_1\delta(\varphi - \vartheta) + p_2\delta(\varphi - (\vartheta + \gamma)) \tag{30}$$

where $0 \leqslant \gamma \leqslant \pi$. This could correspond to a superposition of two phase states with angular distance γ. The usual phase uncertainty is then $\Delta_0\varphi = \sqrt{p_1 p_2}\gamma$, whereas the proposed measure gives

$$\Delta\varphi = \cos^{-1}\sqrt{p_1^2 + 2p_1p_2\cos\gamma + p_2^2}. \tag{31}$$

It can easily be checked that the two results coincide for the case when $p_1 = p_2 = \frac{1}{2}$; the results also approach each other when $\gamma \ll 1$.

Finally, it is interesting to find uncertainty relations between $\Delta\varphi$ and the photon-number uncertainty Δn. These relations are easily obtainable in the Susskind–Glogower model by adding the relations [12]

$$(\Delta n)^2(\Delta \cos\varphi)^2 \geqslant \langle \sin\varphi\rangle^2/4$$
$$(\Delta n)^2(\Delta \sin\varphi)^2 \geqslant \langle \cos\varphi\rangle^2/4. \tag{32}$$

Considering the definition of σ_φ (4), we obtain a relation between the phase dispersion and the photon-number uncertainty (cf [13, 14, 1])

$$(\Delta n)^2(\sigma_\varphi^2 - P_0/2) \leqslant (1 - \sigma_\varphi^2)/4. \tag{33}$$

Here we have used the fact that, in this model, $\langle \hat{\cos}^2\varphi + \hat{\sin}^2\varphi\rangle = 1 - P_0/2$, where P_0 is the vacuum-state probability. Inserting the definition of the phase uncertainty $\Delta\varphi = \sin^{-1}\sigma_\varphi$ into this relation, we arrive at

$$(\Delta n)^2\left[(\tan\Delta\varphi)^2 - \frac{P_0}{2(\cos\Delta\varphi)^2}\right] \geqslant \frac{1}{4}. \tag{34}$$

From this relation we can get a weaker but state-independent relation, the square root of which takes a very simple form

$$\Delta n \tan\Delta\varphi \geqslant \tfrac{1}{2}. \tag{35}$$

Note that for small $\Delta\varphi$ this relation approaches the relation

$$\Delta n \Delta\varphi \geqslant \tfrac{1}{2} \tag{36}$$

the first historical attempt of the number–phase uncertainty relation. On the other hand, when the number uncertainty is very small, the phase uncertainty approaches the value $\pi/2$ and not infinity as required by relation (36).

In this paper, we have seen that the proposed measure of the phase spread (uncertainty) $\Delta\varphi$ has a simple and clear physical meaning. The main advantages of this measure are that: (i) it does not depend on the chosen phase window (as does the standard deviation $\Delta_0\varphi$); (ii) it is possible to find interesting uncertainty relations for this measure; and (iii) it has a physical meaning of angle—in contrast, for example, to the square root of dispersion σ_φ. As has been shown, the properties of the measures $\Delta\varphi$, $\Delta_0\varphi$ and σ_φ are approximately the same for very small $\Delta\varphi$.

Acknowledgments

The author is grateful to V Peřinová and A Lukš for discussion and to the referees for useful comments. This work was supported by MŠMTČR, grant PV 202/1994.

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